Chapter 3 - The simplex method

Recall from chapter 2:

If a L.P. programming problem in standard form has an optimal solution, then there exists a basic feasible solution that is optimal

The simplex method is based on this fact:

- Searches for an optimal solution by moving from one basic feasible solution to another along the edges of the feasible set.
- Eventually, a basic feasible solution is reached at which none of the available edges leads to a cost reduction.
- Such a basic feasible solution is optimal and the algorithm terminates.

Also, recall that we will consider the standard form problem

$$\text{Minimize } c'x$$
$$\text{Subject to } Ax = b$$
$$x \geq 0$$

- And we let P (polyhedron) be the corresponding feasible set.
- We assume that the dimensions of A are $m \times n$
- Rows of A are L. I.
- A_i is the i^{th} column of A
- a_i' is the i^{th} row of A

Notes:
Recall Definition 1.1

3.1 Optimality Conditions

- In L. P. we are minimizing a convex function

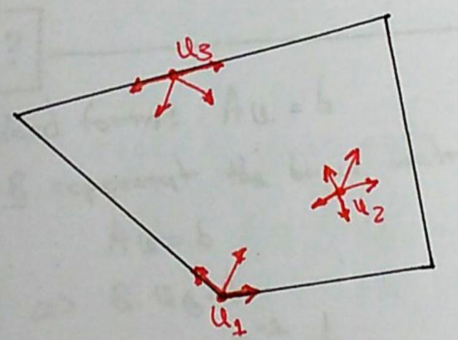
this mean that a local optimal solution is also a global optimal solution

- We will concentrate on the problem of searching for a direction of cost decrease in a neighbourhood of a given basic feasible solution.

$\boxed{Q:}$ $\boxed{\text{Ok, But how should we start looking for a solution?}}$

- Suppose that we are at a point $u \in P$ and that we contemplate moving away from u in the direction of a vector $d \in \mathbb{R}^n$.

  - Clearly: we should only consider those choices of d that do not immediately take us outside the feasible set



Feasible direction at different points of a polyhedron.

Definition 3.1 Let u be an element of a polyhedron P. A vector $d \in \mathbb{R}^n$ is said to be a $\boxed{\text{feasible direction}}$ at u, if there exits a positive scalar θ for which $u + \theta d \in P$

Let: u be a B.F.S

- $B(1), \dots, B(m)$ be the indices of the $\boxed{\text{basic variables}}$

Notes: Basic Variables page 55

If u is a basic solution, the variables $u_{B(1)}, \dots, u_{B(m)}$ are called basic variables

- $B = [A_{B(1)} \dots A_{B(m)}]$ be the corresponding basis matrix.
- Also, $u_i = 0$ for every nonbasic variable

Then the vector $u_B = (u_{B(1)}, \dots, u_{B(m)})$ of basic variables is given by

$$u_B = B^{-1} \cdot b$$

from the standard form !!

Q: Why?

- Standard form: $Au = b$
- Let B represent the basis matrix for ~~basic~~ B.F.S.

$$Au = b$$
$$\Leftrightarrow B \cdot u_B = b$$
$$\Leftrightarrow u_B = B^{-1} \cdot b$$

- Consider the possibility of moving away from u to a new vector $u + \theta d$
- We need to choose a vector d
- Lets choose d by selecting a nonbasic variable u_j (which is initially at zero level) and increasing it to a positive value θ while keeping the remaining nonbasic variables at zero.

- Algebraically: $d_j = 1$ and $d_i = 0$ $(d = (d_1, d_2, \dots, d_j, \dots, d_n)$

- At the same time, vector $\underline{u_B}$ of basic variables changes to $u_B + \theta d_B$ where $d_B = (d_{B(1)}, d_{B(2)}, \dots, d_{B(m)})$ is the vector with those components of $\underline{d}$ that correspond to the basic variables.

- Given that we are only interested in feasible solutions we require:

$$A(u + \theta d) = b$$

- And since $\underline{u}$ is feasible we also have

$$Au = b$$

- For both equality constraints to be satisfied for $\underline{\theta > 0}$

$$\begin{cases} A(u+\theta d) = b \\ Au = b \end{cases} \Rightarrow A.u + \theta Ad = Au \Rightarrow \underset{\text{But this} > 0}{\theta} Ad = 0 \Rightarrow Ad = 0$$

- Then:

$$0 = A.d = \sum_{i=1}^{n} A_i . d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j . d_j$$

Note! This should be 1 so that the full value of θ is accounted for

$$= \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j . 1 = B . d_B + A_j$$

- Since the basis matrix $\underline{B}$ is invertible, we obtain:

$$0 = B.d_B + A_j$$

$$\Leftrightarrow -B . d_B = A_j \Leftrightarrow d_B = -B^{-1}. A_j$$

- The direction vector $\underline{d}$ that we have just constructed will be referred to as the $\underline{j^{th}}$ $\boxed{\text{basic direction}}$

- We have so far guaranteed that the equality constraints are respected as we move away from $\underline{u}$ along the basic direction $\underline{d}$

- But the standard form also includes nonnegativity constraints. What about them.

- Recall that the variable $\underline{u_j}$ is increased, and all other nonbasic variables stay at zero level.
(i.e. they are already nonnegative)

- Thus we only worry about the <u>basic variables</u>

We distinguish two cases:

(a) Suppose that $\underline{u}$ is a nondegenerate B.F.S.

• Then $u_B > 0$ (see definition 2.11):

$$u_B + \theta d_B \geq 0$$

• And feasibility is maintained. $\underline{d}$ is a feasible direction

(b) Suppose $\underline{u}$ is degenerate:

– Then $\underline{d}$ is not always a feasible direction. Why?

– degeneracy implies that there might exist a basic variable $u_{B(i)}$ that is zero.

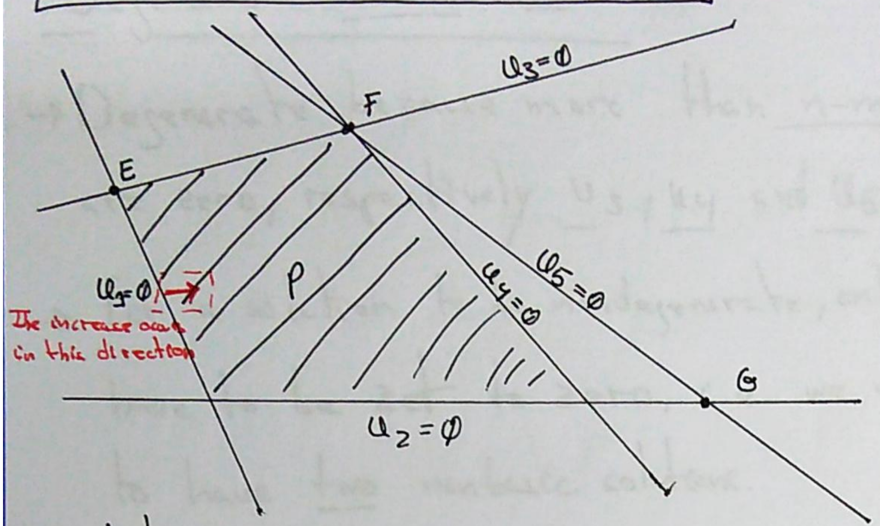
– However, the corresponding direction $\underline{d}$ of the new vector might be negative (because as we have seen in the previous page)

$$d_B = \boxed{-} B^{-1} A_j$$

See here the minus sign!

– Accordingly if we follow the j^{th} basic direction then the nonnegativity constraint for $u_{B(i)}$ is immediately violated.

Q: But what does all of this mean?



- Let $\underline{n=5}$, $\underline{m=3}$, then $\underline{n-m=2}$
- According to Definition 2.11: A vector $\underline{u}$ is degenerate if more than n-m, i.e. $\underline{3}$, components are zero.
- Nondegenerate B.F.S E:
  - Variables u_1 and u_3 are zero (i.e. nonbasic)
  - " u_2, u_4 u_5 are positive basic variables
  - First basic direction is obtained by increasing $\underline{u_1}$ while keeping the other nonbasic variable $\underline{u_3}$ at zero level.
  - This is the direction corresponding to the edge $\underline{EF}$

- Degenerate B.F.S F:

- Degenerate because more than $n-m$ components are zero, respectively u_3, u_4 and u_5
- For a solution to be nondegenerate, only two would have to be set to zero, i.e. we would need to have two nonbasic solutions.
- Let u_3, u_5 be the nonbasic solutions. u_4 would therefore be a basic solution at the zero level.
- A basic ~~solution~~ direction is obtained by increasing u_3 while maintaining u_5 at the zero level.
- This is the direction corresponding to the line FG which takes us outside the feasible set.
- Thus, the basic direction is not a feasible direction.

Lets study the effects of the cost function, if we move along a basic direction:

- If d is the j^{th} basic direction, then the rate $c'd$ of cost change along direction d is given by

$$c'_B \cdot d_B + c_j$$

where $c_B = (c_{B(1)}, \dots, c_{B(m)})$

- But $d_B = -B^{-1}A_j$ which implies:

$$c'_B d_B + c_j = c_j - c'_B B^{-1} A_j$$

Q: Oh... But what does this mean?

Intuitive explantion:

→ c_j is the cost per unit increase in the variable x_j.

→ $-c'_B B^{-1} A_j$ is the cost of compensating change in the basic variables necessitated by the constraint $Ax=b$

Definition 3.2 Let x be a basic solution, let B be an associated basis matrix and let c_B be the vector of costs of the basic variables. For each j, we define the reduced cost $\overline{c_j}$ of the variable x_j according to the formula:

$$\overline{c_j} = c_j - c'_B B^{-1} A_j$$

Example 3.1 Consider the L.P. problem:

Minimize: $c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$

Subject to:
$$u_1 + u_2 + u_3 + u_4 = 2$$
$$2u_1 + 3u_3 + 4u_4 = 2$$
$$u_1, u_2, u_3, u_4 \geq 0$$

- $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{bmatrix}$
- The first two columns are $A_1 = (1, 2)$ and $A_2 = (1, 0)$ which are L. I. Therefore we can choose $\underline{u_1}$ and $\underline{u_2}$ as our basic variables
- Basis matrix $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$
- We set $u_3 = u_4 = 0$ and solve for u_1, u_2
  - $Au = b$ with $u_3, u_4 = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 2 \\ 2 & 0 & 0 & 0 & | & 2 \end{bmatrix} \begin{matrix} x(-2)+ \\ x(-\frac{1}{2}) \end{matrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 2 \\ 0 & -2 & 0 & 0 & | & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & | & 1 \end{bmatrix} x(-1)+;$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & | & 1 \end{bmatrix} \Rightarrow \begin{cases} u_1 = 1 \\ u_2 = 1 \\ u_3 = u_4 = 0 \end{cases}$$

- $u = (1, 1, 0, 0)$ is therefore a nondegenerate $\underline{\text{B. F. S}}$
- A basic direction corresponding to an increase in the nonbasic variable $\underline{u_3}$ is constructed as follows. We have $d_3 = 1$ (so that $\underline{\theta}$ component is obtained) and $d_4 = 0$.

- The direction of change of the basic variables is obtained using $\boxed{d_B = -B^{-1} A_j}$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_{B(1)} \\ d_{B(2)} \end{bmatrix} = d_B = -B^{-1} \boxed{A_3} = -\begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$$

Third column of matrix $\underline{A}$

- The cost of moving along this basic direction is:

$$c'd = c'_B \cdot d_B + c_j$$

$$\Leftrightarrow \quad c'd = -\frac{3}{2} c_1 + \frac{1}{2} c_2 + c_3$$

- Consider now <u>Definition 3.2</u> for the case of a basic variable

- Since $\underline{B}$ is the matrix $[A_{B(1)} \dots A_{B(m)}]$ we have $B^{-1}B = I \Leftrightarrow B^{-1} \cdot [A_{B(1)} \cdots A_{B(m)}] = I$ where I is $m \times m$ identity matrix

- In particular $B^{-1} \cdot A_{B(i)}$ is the i^{th} column of the identity matrix, respectively the i^{th} unit vector $\boxed{e_i}$

- Therefore, for every basic variable $x_{B(i)}$ we have

row vector column vector

$$\bar{c}_{B(i)} = c_{B(i)} - c'_B \boxed{B^{-1} A_{B(i)}} = c_{B(i)} - c'_B e_i$$

$$= c_{B(i)} - c_{B(i)} = 0$$

$\therefore$ The <u>reduced cost</u> of every basic variable is zero.

- Given our interpretation of the reduced costs as rates of cost change along certain directions
- The following result should be intuitive

Theorem 3.1 Consider a basic feasible solution $\underline{u}$ associated with basis matrix $\underline{B}$, and let $\bar{c}$ be the corresponding vector of reduced costs

(a) If $\bar{c} \geq \emptyset$ then $\underline{u}$ is optimal

(b) If $\underline{u}$ is optimal and nondegenerate, then $\bar{c} \geq \emptyset$

$\boxed{\text{Proof}}$

(a) • Assume $\bar{c} \geq 0$

• Let $\underline{Y}$ be an arbitrary feasible solution

• Define $d = y - u$

• Feasibility implies $Au = Ay = b$ (since a B.F.S is a B.S., therefore the equalities should still be observed)

• Therefore $Ad = Ay - Au = b - b = \emptyset$

• The latter equality can be rewritten as

$$\underbrace{B \cdot d_B}_{\text{Basic Solutions}} + \underbrace{\sum_{i \in N} A_i d_i}_{\text{Nonbasic solutions}} = 0$$

- Since $\underline{B}$ is invertible, we obtain

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

Notes:

$$B d_B + \sum_{i \in N} A_i d_i = 0$$

$$\Leftrightarrow B d_B = -\sum_{i \in N} A_i d_i$$

$$\Leftrightarrow d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

- And $c'd = \underbrace{c'_B \cdot d_B}_{\text{cost rate of change of the basic variables}} + \underbrace{\sum_{i \in N} c_i d_i}_{\text{cost rate of change of non basic variables}}$

$$= c'_B\left(\sum_{i \in N} B^{-1} A_i d_i\right) + \sum_{i \in N} c_i d_i$$

$$= \sum_{i \in N} \underbrace{(c_i - c'_B B^{-1} A_i)}_{\overline{c_i} \text{ by definition 3.2}} d_i$$

$$= \sum_{i \in N} \overline{c_i} \, d_i$$

- For any nonbasic index $i \in N$ we must have $u_i = 0$
- Since $\underline{y}$ is feasible, $y_i \geq 0$ (because of the standard form)
- Thus $d_i \geq 0$ (recall that $d = y - u$) and $\overline{c_i} \, d_i \geq 0$ (recall that $\bar{c} \geq 0$)

$\therefore$ $c'(y - u) = c'd \geq 0$ and since $\underline{y}$ was an arbitrary feasible solution, $\underline{u}$ is optimal because the cost of transitioning from $\underline{u}$ to $\underline{y}$ is ≥ 0 which implies that $\underline{u}$ has optimal.

(b) Suppose that $\underline{u}$ is a nondegenerate B.F.S. and that $\boxed{\overline{c_j}}$ < 0 for some j :

reduced cost

- Since the reduced cost of a basic variable is always zero, $\underline{u_j}$ must be a nonbasic variable since its cost $\overline{c_j} < 0$
- Since $\underline{u}$ is nondegenerate, the j^{th} basic direction is a feasible direction of cost decrease (as discussed earlier).
- By moving in that direction, we obtain feasible solutions whose cost is less than that of $\underline{u}$, and $\underline{u}$ is not optimal. Therefore, if $c \geq 0$ $\underline{u}$ is optimal

In order to use Theorem 3.1 we need to satisfy feasibility and nonnegativity of the reduced costs

▨ End of theorem 3.1 proof

Definition 3.3 A basis matrix $\underline{B}$ is said to be $\boxed{\text{optimal}}$ if

(a) $B^{-1} b \geq 0$ and

(b) $\overline{c}' = c' - c'_B B^{-1} A \geq 0'$

$\boxed{\text{Recall}}$ that: $u_B = B^{-1} b$ but according to the standard form $u \geq 0$

∴ If an optimal basis is found, the corresponding basic solution is feasible, satisfies the optimality conditions, and is therefore optimal.

3.2 Development of the simplex method

Objective: how to move to a better B.F.S, whenever a profitable basic direction is discovered

Assume: Every B.F.S. is nondegenerate

Suppose: Suppose that we are at a B.F.S solution x and that we have computed the reduced costs $\bar{c}_j$ of the nonbasic variables.

According to Theorem 3.1 if all $\bar{c}_j$ are nonnegative then we have an optimal solution.

On the other hand, if the reduced cost $\bar{c}_j$ of a nonbasic variable x_j is negative, the j^{th} basic direction d is a feasible solution of cost decrease.

This is the direction obtained by letting $d_j = 1$, $d_i = 0$ $\forall i \neq (B(1), \ldots, B(m)$ and $j)$ and $d_B = -B^{-1}A_j$

While moving along this direction d, the nonbasic variable x_j becomes positive and all other nonbasic variables remain at zero.

Notes: Recall that d_j starts out at zero level and should be set $(d_j = 1)$ so that we have x_j

- We describe this situation by saying that u_j (or A_j) <u>enters</u> or is <u>broght into the basis</u>.

- Once we start moving away from u along the direction d we are tracing points of the form $u + \theta d$, where $\theta \geq 0$

- Since <u>costs decrease</u> along the direction d it is desirable to move as far as possible.

- This takes us to the point $u + \theta^* d$ where

$$\theta^* = \max \{ \theta \geq 0 \mid u + \theta d \in P \}$$

- The resulting cost change is $\underline{\theta^* c' d}$ which is the same as $\theta^* \bar{c}_j$ (recall that $\bar{c}_j$ is the reduced cost <u>per unit</u>, and θ^* just calculate how many units we can go whilst still staying inside the polyhedron)

- We now derive a formula for $\underline{\theta^*}$:
  - Given that $Ad = 0$ we have:

$$\begin{cases} A(u + \theta d) = b \\ Au = b \end{cases} \Rightarrow A(u + \theta d) = b = Au$$
$$\Leftrightarrow \boxed{A(u + \theta d) = Au = b},$$

for all θ and the equality constraints will never <u>be violated</u>.

- Thus $\underline{u + \theta d}$ can become $\boxed{\text{infeasible}}$ only if one of its components becomes $\underline{\text{negative}}$

- We distinguish two cases:

(a) If $d \geq 0$, then $u + \theta d \geq 0 \ \forall \theta \geq 0$, the vector $\underline{u + \theta d}$ never becomes $\boxed{\text{infeasible}}$ and we let $\theta^* = \infty$

(b) If $d_i < 0$ for some $\underline{i}$, the constraint $\underline{u_i + \theta d_i \geq 0}$ becomes $\theta \leq -\frac{u_i}{d}$. This constraint on $\underline{\theta}$ must be satisfied for every i with $\underline{d_i < 0}$. Thus, the largest possible value of $\underline{\theta}$ is:

$$\theta^* = \min_{\{i \mid d_i < 0\}} \left(-\frac{u_i}{d_i} \right)$$

Recall that if $\underline{u_i}$ is a nonbasic variable, then either u_i is the entering variable and $d_i = 1$ or else $d_i = 0$. In either case, $\underline{d_i}$ is nonnegative which invalidates this $\underline{\text{clause}}$

This means that we only need to consider the $\underline{\text{basic variables}}$.

- And we have the equivalent formula

$$\theta^* = \min_{\{i=1,\dots,m) \, d_{B(i)}<0\}} \left(- \frac{u_{B(i)}}{d_{B(i)}} \right)$$

$\theta^* > 0$ because $u_{B(i)} > 0 \ \forall i$ as a consequence of nondegeneracy (see definition 2.11, pp. 59)

<u>Example 3.2</u> This is a continuation of Example 3.1 from the previous section, dealing with the L.P. problem:

Minimize: $c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$

Subject to:
$$u_1 + u_2 + u_3 + u_4 = 2$$
$$2u_1 + 3u_3 + 4u_4 = 2$$
$$u_1, u_2, u_3, u_4 \geq 0$$

- At the time we obtained a <u>basic feasible solution</u> $u = (1, 1, 0, 0)$

- Recall that the reduced cost $\overline{c_3}$ of the nonbasic variable was $-\frac{3}{2} c_1 + \frac{c_2}{2} + c_3$

- Suppose that $c = (2, 0, 0, 0)$ in which case we have $\bar{c}_3 = -\frac{3}{2} \cdot 2 + \frac{0}{2} + 0 \iff \boxed{\bar{c}_3 = -3}$

- Since $\bar{c}_3$ is negative (i.e. the cost is reducing) we form the corresponding basic direction:

Recall that d_3 goes from 0 to 1

$$d = \left(-\frac{3}{2}, \frac{1}{2}, \boxed{1}, 0\right)$$

- Consider vectors of the form $u + \theta d$ with $\theta \geq 0$
    - As θ increases, the only component of u that decrease is the first one (because $d_1 < 0$ and $u \geq 0$).
    - The largest possible value of θ is given by

$$\theta^* = -\left(\frac{u_1}{d_1}\right) = 2/3$$

Notes:

$$\left.\begin{matrix} u_1 = 1 \\ d_1 = -3/2 \end{matrix}\right\} \frac{u_1}{d_1} = -\frac{1}{-\frac{3}{2}} = \frac{2}{3}$$

- This takes us to the point $y = u + \theta^* d$

$$\begin{aligned} y &= u + \theta^* d \\ &= u + 2/3\, d \\ &= (0, 4/3, 2/3, 0) \end{aligned}$$

- Note that the columns $\underline{A_2}$ and $\underline{A_3}$, corresponding to the nonzero variables at the new vector $\underline{y}$ are: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and are L. I.

- Since they are L.I. they form a $\underline{\text{basis}}$ and $\underline{y}$ is a new B. F. S.

- $\underline{\text{In}}$ $\underline{\text{particular}}$:

$$u = (1, 1, \boxed{0}, 0)$$
$$y = (0, \tfrac{4}{3}, \boxed{\tfrac{2}{3}}, 0)$$

Variable $\underline{u_3}$ has $\underline{\text{entered the basis}}$ and variable $\underline{u_1}$ has $\underline{\text{exited the basis}}$

[end of the example] - - - - - - - - - - - -

- Once $\underline{\theta^*}$ is chosen, and assuming it is finite, we move to the new feasible solution $y = u + \theta^* d$

- Since $\underline{u_j = 0}$ (i.e. in the original vector $\underline{u}$) and $d_j = 1$ we have $y_j = \theta^* > 0$

Let $\underline{l}$ be a $\underline{\text{minimizing}}$ index in

(from page 88) $\theta^* = \min\limits_{\{i=1,\dots,m \mid d_{B(i)}<0\}} \left(-\dfrac{u_{B(i)}}{d_{B(i)}} \right)$

i.e.

$$-\frac{u_{B(l)}}{d_{B(l)}} = \min_{\{i=1,\dots,m \mid d_{B(i)}<0\}} \left(-\frac{u_{B(i)}}{d_{B(i)}} \right) = \theta^*$$

- In particular: $d_{B(l)} < 0$ (because of equation 3.2, page 88)

and $\boxed{u_{B(l)} + \theta^* d_{B(l)} = 0}$

$\boxed{\text{Q: Why } u_{B(l)} + \theta^* d_{B(l)} = 0 \text{ ?}}$

- Recall that L.P. in standard form $\begin{matrix} Au = b \\ u \geq 0 \end{matrix}$
- Therefore, the component $\underline{l}$ of the new vector $\underline{y}$, i.e. $u_{B(l)} + \theta^* d_{B(l)}$ should have minimal value
- This implies a minimum value of $\underline{\theta}$ so that the new point is still within the polyhedron

Let $\underline{l}$ be a $\underline{\text{minimizing}}$ index in

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i.e.

$$- \frac{u_{B(l)}}{d_{B(l)}} = \min_{\{i=1,\dots,m \mid d_{B(i)}<0\}} \left(- \frac{u_{B(i)}}{d_{B(i)}} \right) = \theta^*$$

- In particular: $d_{B(l)} < 0$ (because of equation 3.2, page 88)

and $\boxed{u_{B(l)} + \theta^* d_{B(l)} = 0}$

Q: Why $u_{B(l)} + \theta^* d_{B(l)} = 0$?

- Recall that L.P. in standard form $Au = b$, $u \geq 0$
- Therefore, the component $\underline{l}$ of the new vector $\underline{y}$, i.e. $u_{B(l)} + \theta^* d_{B(l)}$ should have minimal value
- This implies a minimum value of $\underline{\theta}$ so that the new point is still within the polyhedron.

$\boxed{Q:}$ **What does $u_{B(\ell)} + \theta^* d_{B(\ell)} = 0$ means?**

- After moving to the new point y:
  - The basic variable $u_{B(\ell)}$ has become zero
  - Whereas the nonbasic variable u_j has now become positive (as seen in example 3.2)
  - This suggests that u_j should replace $u_{B(\ell)}$ in the basis

    → the variable that justified the new direction

  - Accordingly, we take the old basis matrix B and replace $A_{B(\ell)}$ with A_j thus obtaining the matrix

$$\overline{B} = \begin{bmatrix} | & & | & | & | & & | \\ A_{B(1)} & \cdots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \cdots & A_{B(m)} \\ | & & | & | & | & & | \end{bmatrix}$$

Equivalently, we are replacing the set $\{B(1), \dots, B(m)\}$ of basic indices by a new set $\{\bar{B}(1), \dots, \bar{B}(m)\}$ of indices given by

$$\bar{B}(i) = \begin{cases} B(i), & i \neq l \\ j, & i = l \end{cases}$$

Theorem 3.2

(a) The columns $A_{B(i)}$, $i \neq l$ and A_j are L.I. and therefore $\bar{B}$ is a basis matrix

(b) The vector $y = u + \theta^* d$ is a B.F.S. associated with the basis matrix $\bar{B}$

Proof:

(a) If the vectors $A_{\bar{B}(i)}$, $i = 1, \dots, m$ are L.D (Dependent) then there exist coefficients $\lambda_1, \dots, \lambda_m$ not all of them zero, such that

$$\boxed{\sum_{i=1}^{m} \lambda_i A_{\overline{B(i)}} = 0}$$

This is the definition of L.Dependent

which implies that:

$$\sum_{i=1}^{m} \lambda_i \boxed{B^{-1} A_{\bar{B}(i)}} = 0$$

Notes:

In this formula we are just studying what happens when we introduce B^{-1}

and the vectors $\boxed{B^{-1} A_{\bar{B}(i)}}$ are also <u>L. D.</u>

- To show that this <u>is not</u> the case we will prove that the vectors $B^{-1}A_{B(i)}$ $i \neq l$ and $B^{-1}A_j$ are L. I. :

- We have $B^{-1}B = I$
- Since $A_{B(i)}$ is the i^{th} column of $\underline{B}$, it follows that the vectors $B^{-1}A_{B(i)}$, $i \neq l$ are all the unit vectors except for the l^{th} unit vector
- These unit vectors are L.I. and their l^{th} component is <u>zero</u>.
- On the other hand, $\underline{B^{-1}A_j}$ is equal to $\underline{-d_B}$

Notes:

This comes from Equation 3.1, pp 83:

$$d_B = -B^{-1}A_j \iff B^{-1}A_j = -d_B$$

- Its l^{th} entry, $-d_{B(l)}$, is nonzero by the definition of l (since it is a basic variable, thus its direction should have some value)

→ This $B^{-1}A_j$ is L.I. from the unit vectors $B^{-1}A_{B(i)}$, $i \neq l$ since at least one of the entries $\lambda_i \neq 0$, which makes it impossible for the vectors to be L.D.:

$$\sum_{i=1}^{m} \lambda_i B^{-1} A_{\bar{B}(i)} = 0$$

$$\Leftrightarrow \underbrace{\underbrace{\sum_{i \neq j} \left(\lambda_i B^{-1} A_{\bar{B}(i)} \right)}_{\text{Unit vectors therefore the } \lambda_i\text{'s will have to be zero}} + \underbrace{\lambda_j B^{-1} A_{\bar{B}(j)}}_{-d_{B(l)} \neq 0}}_{\text{No way for this to produce a zero vector}} = 0$$

$\therefore B^{-1}A_j$ is L.I. from $B^{-1}A_{B(i)}$

(b) We have $\boxed{y \geq 0}$, $Ay = b$ and $y_i = 0$ for $i \neq \bar{B}(1), \dots, \bar{B}(m)$

Furthermore the columns $A_{\bar{B}(1)}, \dots, A_{\bar{B}(m)}$ have just been shown to be L.I. It follows that y is a B.F.S. associated with matrix $\bar{B}$.

Q: Why is $y \geq 0$?

1) $y = u + \theta^* d$
2) $u \geq 0$ from the standard form
3) $d_j = 1$ implies $y_j = \theta^* > 0$

Therefore $y = \underbrace{u}_{\geq 0} + \underbrace{\theta^* d}_{y_j = \theta^*} \geq 0$

Since θ^* is positive:

- New B.F.S $u + \theta^* d$ is distinct from u
- Since d is a direction of cost decrease:

  Cost of new B.F.S. is strictly smaller
- We have therefore accomplished our goal of moving to a new B.F.S with lower cost

We can now summarize a typical iteration of the simplex method (a.k.a. a $\boxed{\text{pivot}}$)

It is convenient to define a vector $u = (u_1, \dots, u_m)$ by letting

$$u = -d_B = B^{-1} A_j$$

where A_j is the column that enters the basis, in particular $u_i = -d_{B(i)}$ for $i = 1, \dots, m$

An iteration of the simplex method (a.k.a. pivot)

1. Start with $B = [A_{B(1)} \dots A_{B(m)}]$ and let x be a B.F.S.

2. Compute the reduced costs $\overline{c_j}$ for all nonbasic indices j: $\overline{c_j} = c_j - c'_B B^{-1} A_j$

   If $\overline{c} \geq 0$ then: (Theorem 3.1)

   Current B.F.S. is optimal;

   Terminate;

   Else

   Choose some j for which $\overline{c_j} < 0$ (cost decreases)

3. Compute $u = B^{-1} A_j$

   Notes: Recall that A_j is the column that enters the basis

   Notes: This is a condition that needs to hold when we move in direction d_j

   If no component of u is positive:

   Notes: - Page 88, second paragraph, a)

   $\theta^* = \infty$;

   Optimal cost is $-\infty$;

   Algorithm terminates;

4. If some component of u is positive:

   Notes: - Page 88, 2nd paragraph b)

$$\theta^* = \min_{\{i = 1, \dots, m \mid u_i > 0\}} \frac{x_{B(i)}}{u_i}$$

An iteration of the simplex method (a.k.a. pivot)

1. Start with $B = [A_{B(1)} \cdots A_{B(m)}]$ and let $\underline{u}$ be a B.F.S.

2. Compute the reduced costs $\bar{c}_j$ for all nonbasic indices j: $\bar{c}_j = c_j - c_B' B^{-1} A_j$

   If $\bar{c} \geq 0$ then: (Theorem 3.1)

   Current B.F.S. is optimal;

   Terminate;

   Else

   Choose some j for which $\bar{c}_j < 0$ (cost decreases)

3. Compute $u = B^{-1} A_j$

   Notes: Recall that A_j is the column that enters the basis

   Notes: This is a condition that needs to hold when we move in direction d_j

   If no component of $\underline{u}$ is positive:

   Notes: - Page 88, second paragraph, a)

   $\theta^* = \infty$;

   Optimal cost is $-\infty$;

   Algorithm terminates;

4. If some component of $\underline{u}$ is positive:

   Notes: - Page 88, 2nd paragraph b)

$$\theta^* = \min_{\{i = 1, \dots, m \mid u_i > 0\}} \frac{u_{B(i)}}{u_i}$$

5. Let l be such that $\theta = \frac{u_{B(l)}}{u_l}$ (i.e. the minimizing index). Form a new basis by replacing $A_{B(l)}$ with A_j. If y is the new B.F.S the values of the new basic variable are $y_j = \theta^*$ and $y_{B(i)} = u_{B(i)} - \theta^* u_i$, $i \neq l$

$\boxed{Q:}$ But which B.F.S. should be used initially?

A:- Start with an arbitrary B.F.S.
- Which for feasible standard forms is guaranteed to exist

Theorem 3.3 Assume that the feasible set is nonempty and that every basic feasible solution is nondegenerate. Then, the simplex method terminates after a finite number of iterations. At termination, there are the following two possibilities:

(a) We have an optimal basis B and an associated B.F.S. is optimal.

(b) We have found a vector d satisfying $A \cdot d = 0$, $d \geq 0$ and $c'd < 0$ and the optimal cost is $-\infty$

Proof: $\boxed{Q:}$ $\boxed{\text{How can we proof simplex stoppase?}}$

- If the algorithm terminates due to stopping criterion in Step 2:
  - Then the optimality conditions in Theorem 3.1 have been met.
  - B is an optimal basis, and the current B.F.S is optimal

- If the algorithm terminates due to stopping criterion in Step 3:
  - Then we are at a B.F.S. x and have a nonbasic variable x_j such that $\bar{c}_j < 0$ such that the corresponding basic direction d satisfies $Ad = 0$ and $d \geqslant 0$.
  - In particular: $x + \theta d \in P \; \forall \theta > 0$
  - Since $c'd = \bar{c}_j < 0$ by taking θ arbitrarily large, the cost can be made arbitrarily negative and the optimal cost is $-\infty$

- At each iteration: algorithm moves by a positive amount θ^* along a direction d that satisfies $c'd < 0$. Therefore, the cost of every B.F.S. visited is strictly less than the previous one. Since there is a finite number of B.F.S., the algorithm must eventually terminate. ∎

Pivot Selection

- The simplex algorithm has certain degrees of freedom:
  - Step 2: Which j to choose from those where reduced cost $\bar{c}_j$ is negative?;
  - Step 5: There may be several indices l that attain the minimum σ^*. Which one to choose?
- Regarding the choice of j (entering column) the following rules are natural candidates:

  (a) Choose a column A_j with $\bar{c}_j < 0$ whose reduced cost is the most negative.
  Idea: converge to a solution faster.

  (b) Choose a column with $\bar{c}_j < 0$ for which the corresponding cost decrease $\sigma^* |\bar{c}_j|$ is largest.
  Idea: move the furthest along the ~~dir~~ chosen direction.

Notes:
Empirical evidence suggests that the overall running time between a) and b) does not improve.

3.3. Implementations of the simplex method

- It should be clear from the statement of the algorithm that the vectors $B^{-1}A_j$ play a key role.

- If these vectors are available then:
  - the reduced costs;
  - the direction of motion;
  - and the stepsize θ^*.

Can be easily computed.

Notes:

$\bar{c}_j = c_j - c_B' B^{-1} A_j$

$\theta^* = \min_{\{i=1,\dots,m \mid u_i > 0\}} \frac{u_{B(i)}}{u_i}$

If B is an $m \times m$ matrix and $b \in \mathbb{R}^m$ is a given vector then:

- Computing B^{-1} takes $O(m^3)$
- Computing $Bu = b$ takes $O(m^3)$
- Computing $B \cdot b$ takes $O(m^2)$
- Computing inner product $p'b$ of two m-dimensional vector takes $O(m)$

Naive Implementation

- At the beginning of each iteration:
  - Form basis matrix $B = [A_{B(1)} \dots A_{B(m)}]$
  - Compute $p' = c'_B B^{-1}$ by solving $p'B = c'_B$ for the unknown vector p $O(m^3)$
  - Reduced cost of any variable x_j:

$$\bar{c}_j = c_j - \boxed{c'_B B^{-1}} \boxed{A_j}$$
$$= c_j - p' A_j$$

Notes: Notice that we haven't chose a nonbasic variable x_j so we don't have A_j

- $O(mn)$ Depending on the pivot rule we may have to compute all of the reduced costs or we may have to compute one at a time until a variable with a negative reduced cost is encountered

- Once a column A_j is selected we solve
$$Bu = A_j \qquad O(m^3)$$
In order to obtain:
$$u = B^{-1} A_j$$

- At this point:

  We can form the direction along which we will be moving away from the current B.F.S.

- We finally determine θ^* and the variable that will exit the basis.

- We then construct the new B.F.S.

Q: What is the total running time of the naïve implementation?

$$\underbrace{\underbrace{O(m^3)+O(m^3)+O(mn)}_{O(2m^3+mn)}}_{O(m^3+mn)}$$

Q: Can we do better than $O(m^3+mn)$?

Yes we can ⇓

Revised Simplex Method

- Much of the computational effort in the naive approach is due to the need for solving two linear systems of equations

- Alternative implementation: matrix B^{-1} is made available at the beginning of each iteration

  Also, the vectors:

  $$c'_B B^{-1}$$

  $$B^{-1} A_j$$

  Are computed by a matrix-vector multiplication

- For this approach to be practical, we need an efficient method for updating matrix B^{-1} each time we effect a change of basis.

Let

$$B = [\,A_{B(1)} \;\dots\; A_{B(m)}\,]$$

be the basis matrix at the beginning of an iteration and let

The index j enters the basis

$$\bar{B} = [\,A_{B(1)} \cdots A_{B(\ell-1)} \; A_j \; A_{B(\ell+1)} \cdots A_{B(m)}\,]$$

The index ℓ leaves the basis

These two basis matrices have the same columns except for the ℓ^{th} column $\underline{A_{B(\ell)}}$ has been replaced by $\underline{A_j}$

<u>Definition 3.4</u> Given a matrix the operation of adding a constant multiple of one row to the same or to another row is called an <u>elementary row operation</u>

<u>Example 3.3</u>

$$Q = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad QC = \begin{bmatrix} 11 & 14 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

→ Multiply the third row of $\underline{C}$ by $\underline{2}$ and add to the first Row

- Generalizing example 3.3:

Multiplying the j^{th} row by β and adding it to the i^{th} row (for $i \neq j$) is the same as left-multiplying by the matrix

$$\Phi = I + D_{ij}$$

where D_{ij} (is a matrix) $= \begin{cases} 0 & \text{(every thing else)} \\ \beta, i, j & (i^{th} \text{ row and } j^{th} \text{ column}) \end{cases}$

It can be shown that the determinant of such a matrix Φ is 1, therefore the matrix is invertible

Notes:

If $\det(A) = 0 \Rightarrow$ matrix not invertible $\Leftrightarrow \begin{cases} \text{rows of } A \text{ are L.D.} \\ \text{columns of } A \text{ are L.D.} \end{cases}$

If $\det(A) \neq 0 \Rightarrow$ matrix is invertible $\Leftrightarrow \begin{cases} \text{rows of } A \text{ are L.I.} \\ \text{columns of } A \text{ are L.I.} \end{cases}$

- Suppose now that we apply a sequence of k elementary row operations and that the k^{th} such operation corresponds to a left-multiplication by a certain inversible matrix ρ_k.

Then, the sequence of these elementary row operations is the same as left-multiplication by the invertible matrix

$$Q_K Q_{K-1} \cdots Q_2 Q_1$$

This final matrix is invertible

$\therefore$ Performing a sequence of elementary row operations is equivalent to left-multiplying by a certain invertible matrix.

Since $B^{-1}B = I$ we see that $B^{-1}A_{B(i)}$ is the i^{th} unit vector e_i

$$B^{-1}.B = \begin{bmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{l-1} & u & e_{l+1} & \cdots & e_m \\ | & & | & | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_l & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{bmatrix}$$

where $u = B^{-1}A_j$

Notes:
For the next page
Replace the l^{th} column by the l^{th} unit vector

Consider the sequence of elementary row operations that will change the previous matrix to the identity matrix (I):

(a) For each $i \neq l$:

- Add the l^{th} row times $-\frac{u_i}{u_l}$ to the i^{th} row
- This replaces u_i by zero

(b) Divide the l^{th} row by u_l. This replaces u_l by one.

This sequence of elementary row operations is equivalent to left-multiplying $B^{-1}B$ by a certain invertible matrix ϕ that must produce the identity, i.e:

$$\phi B^{-1}\bar{B} = I$$

$$\Leftrightarrow \quad \phi B^{-1} = \bar{B}^{-1}$$

$\therefore$ In order to get $\bar{B}^{-1}$ just start with B^{-1} and apply the sequence of elementary row operations

Example 3.4

$$B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & -2 \end{bmatrix} \quad u = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}$$ Suppose $l = 3$

Objective: Replace u by $e_l = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- Elementary row sequence:
  - Multiply the third row by 2 and add it to the first, i.e.

    $(1, 0, 2)$
  - Subtract the third row from the 2^{nd} row:

    $(0, 1, -1)$
  - Divide the 3^{rd} row by 2:

    $(0, 0, 1/2)$
- Which implies that $Q = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}$

$$Q B^{-1} = \overline{B^{-1}}$$

$$\Leftrightarrow \overline{B^{-1}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -4 & -1 \\ -6 & 6 & 3 \\ 2 & -3/2 & -1 \end{pmatrix}$$

When $\underline{B^{-1}}$ is updated like this we have a method known as $\boxed{\text{revised simplex method}}$

An iteration of the revised simplex method:

① Start with a basis consisting of basic columns $A_{B(1)}, \dots, A_{B(m)}$ an associated $\underline{\text{B.F.S}}$ u and the inverse B^{-1} of the basis matrix

② Compute the row vector $p' = c'_B B^{-1}$ and compute reduced costs $\bar{c}_j = c_j - p'A_j$.

If ~~the~~ reduced costs ≥ 0:

Current $\underline{u}$ is optimal

Terminate;

Else

Choose some $\underline{j}$ for which $\bar{c}_j < 0$

③ Compute $u = B^{-1}A_j$. If no component of $\underline{u}$ is positive

~~4~~ Optimal cost $= -\infty$

Terminate;

④ If some component of $\underline{u}$ is positive:

$$\theta^* = \min_{\{i=1,\dots,m \mid u_i > 0\}} \frac{u_{B(i)}}{u_i}$$

↳

(5.) Let $\underline{l}$ be such that $\theta^* = \underline{\frac{u_{B(l)}}{u(l)}}$. Form a new basis by replacing $\underline{A_{B(l)}}$ with $\overline{A_j}$. If $\underline{y}$ is the new $\underline{B.F.S}$, the values of the new basic variables are:

$$\begin{cases} y_j = \theta^* \\ y_{B(i)} = u_{B(i)} - \theta^* u_i & \forall i \neq l \end{cases}$$

Notes: Concatenate vector u to the end of B^{-1}

(6.) Form the $m \times (m+1)$ matrix $[B^{-1} | u]$.
Add to each one of its rows a multiple of the $l^{\underline{th}}$ row to make the last column equal to the unit vector $\underline{e_l}$. The first $\underline{m}$ columns of the result is the matrix $\overline{B^{-1}}$.

The full tableau implementation

Instead of maintaining and updating the matrix $\underline{B^{-1}}$ we maintain and update the $m \times (n+1)$ matrix

$$B^{-1}[b \mid A]$$

with columns:

$B^{-1}b$
$B^{-1}A_1$
$\vdots$
$B^{-1}A_n$

- This matrix is called the $\boxed{\text{simplex tableau}}$

- Column $B^{-1}b$, called the $\boxed{\text{zeroth column}}$ contains the values of the basic variables

- Column $B^{-1}A_i$ - i^{th} column of the tableau

- Column $u = B^{-1}A_j$ (variable that enters the basis) is called the $\boxed{\text{pivot column}}$

- If the l^{th} basic variable exits the basis, the l^{th} row of the tableau is called the $\boxed{\text{pivot row}}$

- Element belonging to both the pivot row and the pivot column is called the $\boxed{\text{pivot element}}$.

- Note that the pivot element is $\underline{u_l}$ and is always positive.

- The information contained in the rows of the tableau admits the following interpretation. The equality constraints are initially given to us in the form

$$b = Ax$$

- This equality can also be expressed as:

$$B^{-1}b = B^{-1}Au$$ (we just multiplied both sides by B^{-1})

which is precisely the information in the tableau

- In other words, the rows of the tableau provide the coefficients of the equality constraints $B^{-1}b = B^{-1}Au$

- At the end of each iteration:

  Update tableau $B^{-1}[b|A]$

  Compute $\overline{B^{-1}}[b|A]$

- This can be accomplished by left-multiplying the simplex tableau with a matrix Q satisfying $QB^{-1} = \overline{B}^{-1}$. I.e. set of elementary row operations that turns B^{-1} to $\overline{B}^{-1}$.

- That is, we add to each row a multiple of the pivot row to set all entries of the pivot column to zero, with the exception of the pivot element which is set to one (page 97).

- It is customary to augment the simplex tableau by including a top row (zeroth row):

  - Value $-c'_B u_B$ negative of the current cost (minus sign allows for a simple update rule)

  - Row vector of reduced costs

- In more detail:

Current cost	Reduced costs		
$-c'_B u_B$	$\bar{c}_1$	$\dots$	$\bar{c}_n$
$u_{B(1)}$			
$\vdots$	$B^{-1}A_1$	$\dots$	$B^{-1}A_n$
$u_{B(m)}$			

An iteration of the full tableau implementation

① Start with the tableau associated with basis matrix B and the corresponding B.F.S. u

② Examine the reduced costs in the zeroth row.

If they are nonnegative:

Current B.F.S. is optimal

Terminate

Else:

Choose some j for which $\bar{c}_j < 0$

③ Consider the vector $u = B^{-1}A_j$ (j^{th} column of tableau)

If no component of u is positive:

Optimal cost is $-\infty$

Terminate;

④ For each i for which u_i is positive:

Compute $\dfrac{u_{B(i)}}{u(i)}$

Let l be the index of the smallest ratio.

Column $A_{B(l)}$ exits and column A_j enters the basis.

⑤ Add to each row of the tableau a constant multiple of the l^{th} row (pivot row) so that u_l (the pivot element) becomes one and all other entries of the pivot column become zero.

③ Consider the vector $\underline{u = B^{-1} A_j}$ ($j^{\underline{th}}$ column of tableau)

If no component of $\underline{u}$ is positive:

Optimal cost is $-\infty$

Terminate;

④ For each $\underline{i}$ for which $\underline{u_i}$ is $\underline{\text{positive}}$:

Compute $\dfrac{u_{B(i)}}{u(i)}$

Let $\underline{l}$ be the $\underline{\text{index}}$ of the smallest ratio.

Column $A_{B(l)}$ $\underline{\text{exits}}$ and column $\underline{A_j}$ $\underline{\text{enters}}$ the basis.

⑤ Add to each row of the tableau a constant multiple of the $l^{\underline{th}}$ row ($\underline{\text{pivot row}}$) so that $\underline{u_l}$ (the pivot element) becomes one and all other entries of the $\underline{\text{pivot column}}$ become zero.

Example 3.5 Consider the problem:

Minimize: $-10u_1 - 12u_2 - 12u_3$

Subject to:

$R_1: \quad u_1 + 2u_2 + 2u_3 \leq 20$

$R_2: \quad 2u_1 + u_2 + 2u_3 \leq 20$

$R_3: \quad 2u_1 + 2u_2 + u_3 \leq 20$

$u_1, u_2, u_3 \geq 0$

First, lets try to visualize the feasible set

→ $u_1 + 2u_2 + 2u_3 = 20$

u_1u_2-plane:	u_1u_3-plane:	u_2u_3-plane:
$u_1 = 20 - u_2$	$u_1 = 20 - 2u_3$	$\boxed{u_2 = 10 - u_3}$

→ $2u_1 + u_2 + 2u_3 = 20$

u_1u_2-plane:	u_1u_3-plane:	u_2u_3-plane:
$u_1 = 10 - \frac{u_2}{2}$	$\boxed{u_1 = 10 - u_3}$	$u_2 = 20 - 2u_3$

→ $2u_1 + 2u_2 + u_3 = 20$

u_1u_2-plane:	u_1u_3-plane:	u_2u_3-plane:
$\boxed{u_1 = 10 - u_2}$	$u_1 = 10 - u_3/2$	$u_2 = 10 - u_3/2$

- Notice how there are multiple lines that we would need to draw, this would make for a mess of a drawing.

However, all the restrictions need to be observed simultaneously:

- Notice that all the restrictions are inequalities I.e. $\leq$ and $\geq$
- This means that we need to look for the "smaller" lines on each plane, respectively:

$$u_1 u_2\text{-plane}: \; u_1 = 10 - u_2$$

$$u_1 u_3\text{-plane}: \; u_1 = 10 - u_3$$

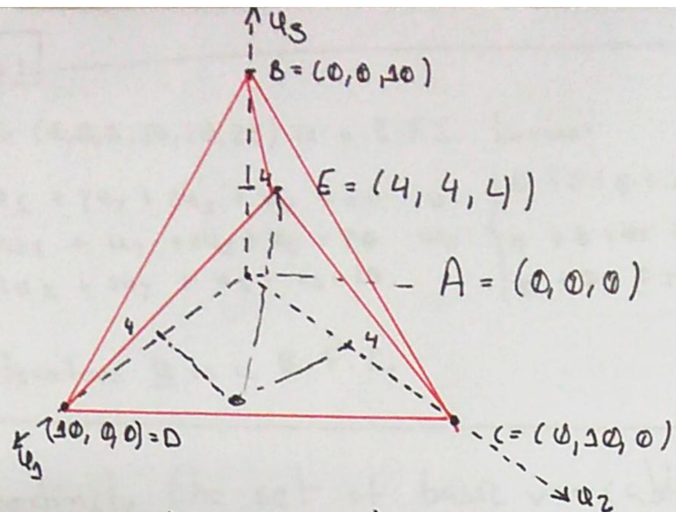
$$u_2 u_3\text{-plane}: \; u_2 = 10 - u_3$$

- Furthermore, we need to check when all the three restrictions (not only the plane projections) are observed:

$$\times(-2)+; \left[\begin{array}{ccc|c} 1 & 2 & 2 & 20 \\ 2 & 1 & 2 & 20 \\ 2 & 2 & 1 & 20 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 20 \\ 0 & -3 & -2 & -20 \\ 0 & -2 & -3 & -20 \end{array}\right] \times\left(-\tfrac{1}{3}\right)+ \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 20 \\ 0 & 1 & 2/3 & 20/3 \\ 0 & -2 & -3 & -20 \end{array}\right] \times 2+; \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 20 \\ 0 & 1 & 2/3 & 20/3 \\ 0 & 0 & -5/3 & -20/3 \end{array}\right] \times\left(-\tfrac{3}{5}\right)+$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 20 \\ 0 & 1 & 2/3 & 20/3 \\ 0 & 0 & 1 & 4 \end{array}\right] \times\left(-\tfrac{2}{3}\right)+ \; \times(-2)+; \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore (4, 4, 4)$ is the intersection point



There exist five <u>extreme points</u>:

$A = (0,0,0) \Rightarrow$ with cost 0

$B = (0,0,10) \Rightarrow$ " " -120

$C = (0,10,0) \Rightarrow$ " " -120

$D = (10,0,0) \Rightarrow$ " " -100

$\boxed{E = (4,4,4) \Rightarrow \text{" " } -136}$ Optimal solution (lower cost)

After introducing <u>slack</u> variables (since in the standard form <u>$Au = b$</u> and <u>not</u> $Au \leq 20$):

$$\text{Minimize:} \quad -10u_1 - 12u_2 - 12u_3$$

$$\begin{aligned}
\text{Subject to:} \quad & u_1 + 2u_2 + 2u_3 + u_4 = 20 \\
& 2u_1 + u_2 + 2u_3 + u_5 = 20 \\
& 2u_1 + 2u_2 + u_3 + u_6 = 20 \\
& u_1, \dots, u_6 \geq 0
\end{aligned}$$

Note that $u = (0,0,0,20,20,20)$ is a <u>B.F.S</u> and can be used to start the ~~solution~~ algorithm.

Notes:

$u = (0,0,0,20,20,20)$ is a B.F.S. because

$$\begin{cases} u_1 + 2u_2 + 2u_3 + u_4 = 20 \\ 2u_1 + u_2 + 2u_3 + u_5 = 20 \\ 2u_1 + 2u_2 + u_3 + u_6 = 20 \end{cases} \overset{u}{\Rightarrow} \begin{cases} 0 + 0 + 0 + 20 = 20 \\ 0 + 0 + 0 + 20 = 20 \\ 0 + 0 + 0 + 20 = 20 \end{cases} \Leftrightarrow \begin{cases} 20 = 20 \\ 20 = 20 \\ 20 = 20 \end{cases}$$

$\therefore$ Therefore $\underline{u}$ is a B.F.S.

Accordingly the set of basic variables is:

$$u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 20 \\ 20 \\ 20 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Nonbasic Variables} := \{u_1, u_2, u_3\} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Basic Variables} := \{u_4, u_5, u_6\} \end{matrix}$$

Accordingly: $B(1) = 4$ (i.e. 4^{th} column of A, respectively $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$)
$B(2) = 5$ (" 5^{th} " " ", " $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$)
$B(3) = 6$ (" 6^{th} " " A, " $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$)

Therefore: $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Costs of the basis matrix: $\left.\begin{matrix} u_4 = 0 \\ u_5 = 0 \\ u_6 = 0 \end{matrix}\right]$ since they do not exist in the cost function $\Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$

Therefore: $c'_B u_B = 0$

Also: $\overline{c'} = c' - c'_B B^{-1} A = c' - \underline{0}\, B^{-1} A = c'$

$\therefore \overline{c'} = c'$

- We can now obtain the <u>tableau</u>:

$-c_B' u_B$	$\bar{c}_1 \quad \dots \quad \bar{c}_n$
$u_{B(1)}$ $\vdots$ $u_{B(m)}$	$B^{-1}A_1 \quad \dots \quad B^{-1}A_n$

Zeroth column

Initial cost

		u_1	u_2	u_3	u_4	u_5	u_6
	0	-10	-12	-12	0	0	0
$u_4 =$	20	1	2	2	1	0	0
$u_5 =$	20	2	1	2	0	1	0
$u_6 =$	20	2	2	1	0	0	1

zeroth row

Basic Variables

Q: Are all the reduced costs in the <u>zeroth</u> row nonnegative?

- No! Therefore we need to choose some $\underline{j}$ for which $\bar{c}_j < 0$.
- Let u_1 be the variable that enters the basis (since $\bar{c}_1 = -10 < 0$)

- Pivot column $u = B^{-1} A_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\begin{array}{l} \overset{?}{>} 0 : \text{Yes} - \\ \overset{?}{>} 0 : \text{Yes} - \\ \overset{?}{>} 0 : \text{Yes} - \end{array}$ $\boxed{\begin{array}{l} \text{Index } 1 \\ \text{Index } 2 \\ \text{Index } 3 \end{array}}$

- Ratios $\dfrac{u_{B(i)}}{u(i)}$, $\boxed{i = 1, 2, 3}$

$$\frac{u_{B(1)}}{u(1)} = \frac{u_4}{u(1)} = \frac{20}{1} = 20$$

$$\left.\begin{array}{l} \dfrac{u_{B(2)}}{u(2)} = \dfrac{u_5}{u(2)} = \dfrac{20}{2} = 10 \\[2ex] \dfrac{u_{B(3)}}{u(3)} = \dfrac{u_6}{u(3)} = \dfrac{20}{2} = 10 \end{array}\right]$$

Two indices have the smallest ratio.
Break the tie:
$l = 2$

- Pivot row index: ~~4~~ $\boxed{2}$
  Pivot column index: 1

  Pivot element $= 2$
  Row=2, column=1

- Column $A_{B(l)} = A_{B(2)} = A_5$ exits the basis

  Column A_1 enters the basis

  $\bar{B}(1) = B(1) = 4$ (i.e. is maintained)

  $\bar{B}(2) = j = 1$ (enters the basis)

  $\bar{B}(3) = B(3) = 6$ (i.e. is maintained)

$$\begin{array}{r c}
 & \\
 & \\
\times(5)+ & \\
 &
\end{array}
\begin{pmatrix}
0 & -10 & -12 & -12 & 0 & 0 & 0 \\
20 & 1 & 2 & 2 & 1 & 0 & 0 \\
20 & \boxed{2} & 1 & 2 & 0 & 1 & 0 \\
20 & 2 & 2 & 1 & 0 & 0 & 1
\end{pmatrix}$$

$\square$:= Pivot element

Perform operations so that the pivot element u_ℓ becomes one and all other entries of the pivot column become zero

• Multiply pivot row by 5 and add to zeroth row:

$$\begin{array}{r c}
 & \\
 & \\
\times(\frac{1}{2})+ & \\
 &
\end{array}
\begin{pmatrix}
100 & 0 & -7 & -2 & 0 & 5 & 0 \\
20 & 1 & 2 & 2 & 1 & 0 & 0 \\
20 & 2 & 1 & 2 & 0 & 1 & 0 \\
20 & 2 & 2 & 1 & 0 & 0 & 1
\end{pmatrix}$$

• Multiply pivot row by $\left(-\frac{1}{2}\right)$ and add to 1st row:

$$\begin{array}{r c}
 & \\
 & \\
 & \\
\times(-1)+ &
\end{array}
\begin{pmatrix}
100 & 0 & -7 & -2 & 0 & 5 & 0 \\
10 & 0 & -3/2 & 1 & 1 & -1/2 & 0 \\
20 & 2 & 1 & 2 & 0 & 1 & 0 \\
20 & 2 & 2 & 1 & 0 & 0 & 1
\end{pmatrix}$$

• Multiply pivot row by (-1) and add to 3rd row:

$$\begin{array}{r c}
 & \\
 & \\
\times(-\frac{1}{2}) & \\
 &
\end{array}
\begin{pmatrix}
100 & 0 & -7 & -2 & 0 & 5 & 0 \\
10 & 0 & -3/2 & 1 & 1 & -1/2 & 0 \\
20 & 2 & 1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & 1
\end{pmatrix}$$

• Multiply pivot row by $\left(-\frac{1}{2}\right)$

$$\begin{pmatrix}
100 & 0 & -7 & -2 & 0 & 5 & 0 \\
10 & 0 & -3/2 & 1 & 1 & -1/2 & 0 \\
10 & 1 & 1/2 & 1 & 0 & 1/2 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & 1
\end{pmatrix}$$

This leads us to a new tableau:

Cost has been reduced to -100	u_1	u_2	u_3	u_4	u_5	u_6
100	0	-7	-2	0	5	0
$u_4 =$ 10	0	1.5	1	1	-0.5	0
$u_1 =$ 10	1	0.5	1	0	0.5	0
$u_6 =$ 0	0	1	-1	0	-1	1

New B.F.S. $u = (10, 0, 0, 10, 0, 0)$

- 1^{st} iteration is over
- 2^{nd} iteration starts:

- Variable u_2 and u_3 have negative costs, respectively -7 and -2.
- Let u_3 enter the basis

$$u = B^{-1}A_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$ indexes that are positive 1 and 2

- Compute ratios: $\dfrac{u_{B(i)}}{u(i)}$ $\quad i = \underbrace{1, 2}$

$$\frac{u_{B(1)}}{u(1)} = \frac{u_4}{u(1)} = \frac{10}{1} = 10$$

$$\frac{u_{B(2)}}{u(2)} = \frac{u_1}{u(2)} = \frac{10}{1} = 10$$

Again tie

Choose $l = 1$

- $A_{B(\ell)} = A_{B(1)} = A_4$ exits the basis

$A_j = A_3$ enters the basis $\Rightarrow \begin{cases} \bar{B}(1) = 3 \\ \bar{B}(2) = B(2) = 1 \\ \bar{B}(3) = B(3) = 6 \end{cases}$

- $\left.\begin{array}{l} \text{Pivot column} = 3 \\ \text{Pivot row} = 1 \end{array}\right\}$ Pivot Element $= 1$ $(1, 3)$

- ~~New~~ tableau :

$$
\begin{array}{l} 1) \times (+2) + j \\ 2) \times (-1) + j \\ 3) \times (1) + j \end{array}
\begin{pmatrix}
100 & 0 & -7 & -2 & 0 & 5 & 0 \\
10 & 0 & 1.5 & \boxed{1} & 1 & -0.5 & 0 \\
10 & 1 & 0.5 & 1 & 0 & 0.5 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & 1
\end{pmatrix}
$$

Pivot element should be one, all other elements in the pivot column should be zero

$$
\rightarrow \begin{pmatrix}
120 & 0 & -4 & 0 & 2 & 4 & 0 \\
10 & 0 & 1.5 & 1 & 1 & -0.5 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 \\
10 & 0 & 2.5 & 0 & 1 & -1.5 & 1
\end{pmatrix}
$$

Cost has been reduced to -120

$$
\begin{array}{c|c|cccccc}
 & & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
 & 120 & 0 & -4 & 0 & 2 & 4 & 0 \\
\hline
u_3 & 10 & 0 & 1.5 & 1 & 1 & -0.5 & 0 \\
u_1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\
u_6 & 10 & 0 & 2.5 & 0 & 1 & -1.5 & 1
\end{array}
$$

new B.F.S. u :

$(0, 0, 10, 0, 0, 10)$

- End of 2^{nd} iteration

Start of 3rd iteration: $u = (0, 0, 10, 0, 0, 10)$

- u_2 is the only variable with negative cost (i.e. -4)

- $u = B^{-1} A_2 = \begin{pmatrix} 1.5 \\ -1 \\ 2.5 \end{pmatrix}$ Positive indexes $\underline{1}$ and $\underline{3}$

- Calculate ratios: $\dfrac{u_{B(i)}}{u_i} \quad i = 1, 3$

- $\dfrac{u_{B(1)}}{u_1} = \cancel{\dfrac{10}{1.5}} = \dfrac{u_3}{u_1} = \dfrac{10}{3/2} = \dfrac{20}{3}$

- $\dfrac{u_{B(3)}}{u_3} = \dfrac{u_6}{u_3} = \dfrac{10}{5/2} = \dfrac{20}{5} = 4$

$\therefore l = 3$

- $A_{B(l)} = A_{B(3)} = A_6$ exits the basis
  A_2 enters the basis

$$\left.\begin{array}{l} \overline{B(1)} = B(1) = 3 \\ \overline{B(2)} = B(2) = 1 \\ \overline{B(3)} = 2 \end{array}\right.$$

- $\left.\begin{array}{l}\text{Pivot column} = 2 \\ \text{Pivot row} = 3\end{array}\right\}$ Pivot Element $= \boxed{2.5}$ $(3, 2)$

$$\begin{array}{l} \\ 3) \times \left(\frac{4}{2.5}\right) \\ 2) \times \left(-\frac{1.5}{2.5}\right) + \\ 1) \times \left(\frac{1}{2.5}\right) + \end{array} \begin{pmatrix} 130 & 0 & -4 & 0 & 2 & 4 & 0 \\ 10 & 0 & 1.5 & 1 & 1 & -0.5 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 10 & 0 & \boxed{2.5} & 0 & 1 & -1.5 & 1 \end{pmatrix}$$

$\boxed{}$:= Pivot Element should be $\underline{\text{one}}$, all other elements in the pivot column should be $\underline{\text{zero}}$

$$\rightarrow \begin{pmatrix} 136 & 0 & 0 & 0 & 3.6 & 1.6 & 1.6 \\ 4 & 0 & 0 & 1 & 0.4 & 0.4 & -0.6 \\ 4 & 1 & 0 & 0 & -0.6 & 0.4 & 0.4 \\ 4 & 0 & 1 & 0 & 0.4 & -0.6 & 0.4 \end{pmatrix}$$

Cost has been reduced to -136

Tableau:

		u_1	u_2	u_3	u_4	u_5	u_6
	136	0	0	0	3.6	1.6	1.6
$u_3 =$	4	0	0	1	0.4	0.4	-0.6
$u_1 =$	4	1	0	0	-0.6	0.4	0.4
$u_2 =$	4	0	1	0	0.4	-0.6	0.4

- All the costs are positive: Algorithm terminates
- Optimal $\underline{u}$: $(4, 4, 4, 0, 0, 0)$
- In the original dimensions (<u>before</u> slack variables)

$$u = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \text{ with cost } 136$$

Example 3.6

Consider a problem described in terms of the following initial tableau:

Basic Variables		u_1	u_2	u_3	u_4	u_5	u_6	u_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$u_5 =$	0	$\boxed{1/4}$ Pivot Element	-8	-1	9	1	0	0
$u_6 =$	0	$1/2$	-12	$-1/2$	3	0	1	0
$u_7 =$	1	0	0	1	0	0	0	1

Q: What is the initial B.F.S. u? $u = (0,0,0,0,0,0,1)$

Use the following pivoting rules:

(a) Select a non basic variable with the most negative reduced cost $\bar{c}_j$ to be the one that enters the basis

(b) Out of the basic variables that are eligible to exit the basis, select the one with the smallest subscript

[resolution :]

①

		u_1	u_2	u_3	u_4	u_5	u_6	u_7
	3	0	-4	$-1/2$	33	3	0	0
u_1	0	1	-32	-4	36	4	0	0
u_6	0	0	$\boxed{4}$ Pivot Element	$3/2$	-15	-2	1	0
u_7	1	0	0	1	0	0	0	1

B.F.S. $u = (0,0,0,0,0,0,1)$

②

	3	0	0	-2	18	1	1	0
$u_1 =$	0	1	0	$\boxed{8}$	-84	-12	8	0
$u_2 =$	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0
$u_7 =$	1	0	0	1	0	0	0	1

B.F.S $u = (0,0,0,0,0,0,1)$

③

	3	$1/4$	0	0	-3	-2	[illegible]	3	0
$u_3 =$	0	$1/8$	0	1	$-21/2$	$-3/2$	[illegible]	1	0
$u_4 =$	0	$-3/64$	1	0	$\boxed{3/16}$ Pivot Element	$1/16$		$-1/8$	0
$u_7 =$	1	$-1/8$	0	0	$21/2$	$3/2$		-1	1

B.F.S $u = (0,0,0,0,0,0,1)$

④

	3	$-1/2$	16	0	0	-1	1	0
$u_3 =$	0	$-5/2$	56	1	0	$\boxed{2}$ Pivot Element	-6	0
$u_4 =$	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0
$u_7 =$	1	$5/2$	-56	0	0	-2	6	1

B.F.S $u = (0,0,0,0,0,0,1)$

5

	3	$-7/4$	44	$1/2$	0	0	-2	0
$u_5 =$	0	$-5/4$	28	$1/2$	0	1	-3	0
$u_4 =$	0	$1/6$	-4	$-1/6$	1	0	$\boxed{1/3}$ Pivot Element	0
$u_7 =$	1	0	0	1	0	0	0	1

B.F.S. $u = (0,0,0,0,0,0,1)$

6

3	$-3/4$	20	$-1/2$	6	0	0	0
0	$1/4$	-8	-1	9	1	0	0
0	$1/2$	-12	$-1/2$	3	0	1	0
1	0	0	1	0	0	0	1

B.F.S. $u = (0,0,0,0,0,0,1)$

Q: Can you see what just happened?

- After six pivots, we have the <u>same basis</u> and the <u>same tableau</u> that we started with
- At each basis change we had $\theta^* = 0$

Q: What does this mean, i.e. that $\theta^* = 0$?

- Recall that we start at B.F.S. $\underline{u}$
- And choose a ~~...~~ nonbasic variable for direction $\underline{d}$
- Then we advance θ^* in that direction
- I.e. $y = u + \theta^* d \overset{\theta^*=0}{\Longleftrightarrow} y = u$
- Therefore the the B.F.S. is never changing and the cost is always the same, respectively, $\underline{3}$

$\therefore_1$ Same sequence of pivots is <u>always</u> repeated

$\therefore_2$ <u>Algorithm never ends.</u>

Comparison of the full tableau and the revised methods

$\boxed{Q_1:}$ $\boxed{\text{How do the revised and tableau methods compare?}}$

$\boxed{Q_2:}$ What are then the time/space requirements for the revised simplex? And for the tableau?

Q_1: Pretend, that the problem is changed to

$$\begin{aligned} \text{Minimize: } & c'u + 0'y \\ \text{Subject to: } & Au + Iy = b \\ & u, y \geq 0 \end{aligned}$$

For this formulation to work with simplex:

1) None of the components of y can become basic
2) I.e. perform basis changes as if y is absent

The vector of reduced cost in the augmented problem

$$[c' \mid 0'] - c_B' B^{-1} [A \mid I] =$$

$$= [\underbrace{c' - c_B' B^{-1} A}_{\bar{c}'} \mid 0' - c_B' B^{-1} I]$$

$$= [\quad \bar{c}' \quad \mid -c_B' B^{-1}]$$

↳

Thus the simplex tableau for the augmented problem takes the form: (see page 99 of the book)

$$\begin{array}{ccc} -c_B' B^{-1} b & \bar{c}' & -c_B' B^{-1} \\ B^{-1} b & B^{-1} A & \boxed{B^{-1}} \end{array}$$

Notes: $\to B^{-1} : I$

- By following the mechanics of the full tableau on the above tableau, the inverse basis matrix B^{-1} is made available at each iteration.
- We can now think of the revised simplex method as being essentially the same as the full tableau method applied to the above augmented problem, except the part of the tableau containing $B^{-1}A$ is never formed explicitly (see page 96 of the book)
- Instead, the pivot column $B^{-1}A_j$ is computed on the fly.

$\therefore$ The revised simplex method is just a variant of the full tableau method.

End of Q_4

Q2:

	Full Tableau	Revised simplex
Memory	$O(m.n)$	$O(m^2)$
Worst-case time	$O(m.n)$	$O(m.n)$
Best-case time	$O(m.n)$	$O(m^2)$

Notes:

- When $n \simeq m$ the algorithm take same time/space
- However, in practice:
  - n is often much larger than m;
  - This favours the revised simplex

3.4 Anticycling: Lexicography and Bland's rule

Lexicography: pivoting rules that prevents the simplex method from cycling (see Example 3.6)

Definition 3.5 A vector $u \in \mathbb{R}^n$ is said to be lexicographically larger (or smaller) than another vector $v \in \mathbb{R}^n$ if $u \neq v$ and the first nonzero component of $u - v$ is positive (or negative, respectively). ~~Symbolic~~ Symbolically we write

$$u \overset{L}{>} v \quad \text{or} \quad u \overset{L}{<} v$$

Example:

$(0,2,3,0) \overset{L}{>} (0,2,1,4)$ $\quad (0,2,3,0) - (0,2,1,4) = (0,0,\boxed{2},-4)$

$(0,4,5,0) \overset{L}{<} (1,2,1,2)$ $\quad (0,4,5,0) - (1,2,1,2) = (\boxed{-1},2,4,-2)$

Lexicographic pivoting rule

1. Choose an entering column A_j arbitrarly, as long as its reduced cost is negative. Let $u = B^{-1}A_j$ be the j^{th} column of the tableau

2. For each i with $u_i > 0$, divide the i^{th} row of the tableau by u_i and choose the lexicographically smallest row. If row l is lexicographically smallest, then the l^{th} basic variable $u_{B(l)}$ exits the basis

Example 3.7 Consider the following tableau (zeroth row is omitted) and suppose that the pivot column is the third one $(j = 3)$

$$\begin{array}{c|cccc} & & & u & \\ 1 & 0 & 5 & 3 & \cdots \\ 2 & 4 & 6 & -1 & \cdots \\ 3 & 0 & 7 & 9 & \cdots \end{array}$$

$$\theta^* = \min\left\{\frac{u_{B(1)}}{u_1}, \frac{u_{B(3)}}{u_3}\right\} = \min\left\{\frac{1}{3}, \frac{3}{9}\right\} = \underbrace{\min\left\{\frac{1}{3}, \frac{1}{3}\right\}}_{\text{tie !!}}$$

Apply lexicographic rule:

$$\begin{array}{c|cccc} 1/3 & 0 & 5/3 & 1 & \cdots \\ 2 & 4 & 6 & -1 & \cdots \\ 1/3 & 0 & 7/3 & 3 & \cdots \end{array}$$

$(1/3, 0, 5/3, 1, \ldots) - (1/3, 0, 7/3, 3, \ldots)$

$= (0, 0, 8/3, 1, \ldots) \Rightarrow$ row 3 $<$ row 1

$\Rightarrow$ Variable $u_{B(3)}$ exits the basis !!

Theorem 3.4 Suppose that the simplex algorithm starts with all the rows in the simplex tableau, other than the zeroth row, lexicographically positive. Suppose that the lexicographic rule is followed. Then:

Notes: $u > 0$

(a) Every row of the simplex tableau, other than the zeroth row, remains lexicographically positive througho the algorithm.

(b) The zeroth row strictly increases lexicographically at each iteration.

(c) The simplex method terminates after a finite number of iterations

Proof is in page 110 of the book ;)

Bland's rule

1. Find the smallest j for which the reduced cost $\bar{c}_j$ is negative and have column A_j enter the basis.

2. Out of all variables u_i that are tied in the le for choosing an exiting variable, select the one wi the smallest subscript i.

3.5 Finding an initial basic feasible solution

Example 3.8: Consider the L.P. problem

Minimize: $u_1 + u_2 + u_3$

Subject to:
$$\begin{aligned} u_1 + 2u_2 + 3u_3 \quad &= 3 \\ -u_1 + 2u_2 + 6u_3 \quad &= 2 \\ 4u_2 + 9u_3 \quad &= 5 \\ 3u_3 + u_4 &= 1 \end{aligned}$$

$$u_1, \dots, u_4 \geq 0$$

$\boxed{Q:}$ What initial B.F.S should be used?

- Difficult to see, right?
- We need to transform the problem:

Minimize: $u_5 + u_6 + u_7 + u_8$

Subject to:
$$\begin{aligned} u_1 + 2u_2 + 3u_3 \quad + u_5 \quad &= 3 \\ -u_1 + 2u_2 + 6u_3 \quad + u_6 \quad &= 2 \\ 4u_2 + 9u_3 \quad + u_7 \quad &= 5 \\ 3u_3 + u_4 \quad + u_8 &= 1 \end{aligned}$$

- Now finding a B.F.S. is trivial

$$u = (0, 0, 0, 0, 3, 2, 5, 1)$$

- Corresponding basic matrix $\underline{B}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- $C_B = (1, 1, 1, 1)$

- $C' = \underline{0}$ (zero vector)

- $\underline{\text{Reduced cost}}$: $c' - c'_B B^{-1} A = 0 - c'_B I A = -c'_B A$

- $-c'_B u_B = -(1,1,1,1) \cdot \begin{pmatrix} 3 \\ 2 \\ 5 \\ 1 \end{pmatrix} = -(3+2+5+1) = -1$

①

		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	-11	0	-8	-21	-1	0	0	0	0
u_5	3	1	2	3	0	1	0	0	0
u_6	2	-1	2	6	0	0	1	0	0
u_7	5	0	4	9	0	0	0	1	0
u_8	1	0	0	3	$\boxed{1}$	0	0	0	0

u_4 enters b

u_8 exits b

②

		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	-10	0	-8	-18	0	0	0	0	0
u_5	3	1	2	3	0	1	0	0	0
u_6	2	-1	2	6	0	0	1	0	0
u_7	5	0	4	9	0	0	0	1	0
u_4	1	0	0	$\boxed{3}$	1	0	0	0	1

u_3 enters

u_4 exits

③

		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	-4	0	-8	0	6	0	0	0	7
u_5	2	1	2	0	-1	1	0	0	-1
u_6	0	-1	$\boxed{2}$	0	-2	0	1	0	-2
u_7	2	0	4	0	-3	0	0	1	-3
u_4	1/3	0	0	1	1/3	0	0	0	1/3

u_2 enters

u_6 exits

④

		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	-4	-4	0	0	-2	0	4	0	-1
u_5	2	$\boxed{2}$	0	0	1	1	-1	0	1
u_2	0	-1/2	1	0	-1	0	1/2	0	-1
u_7	2	2	0	0	1	0	-2	1	1
u_3	1/3	0	0	1	1/3	0	0	0	1/3

5

		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	0	0	0	0	0	2	2	0	1
u_1	1	1	0	0	1/2	1/2	-1/2	0	1/2
u_2	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
u_7	0	0	0	0	0	-1	-1	1	0
u_3	1/3	0	0	1	1/3	0	0	0	1/3

Cost has dropped to zero: We have a feasibl solution to the original problem (recall that the cost was $u_1+u_2+u_3$)

However: artificial variable u_7 is still in basis

Note that ~~the~~ row associated with u_7 is

$$(\underbrace{0, 0, 0, 0}_{\text{Original variables are set to zero...}}, 0, -1, -1, 1, 0)$$

→ This indicates that A has L.D. rows, th This row is redundant and can be rem

→ Also, the artificial variables can be remove

– This leaves us with the following tableau

		u_1	u_2	u_3	u_4
		0	0	0	0
u_1	1	1	0	0	1/2
u_2	1/2	0	1	0	-3/4
u_3	1/3	0	0	1	1/3

– We may now compute the reduced costs of the original variables (fill in the zeroth row of tableau

– And start executing the simplex method on the original problem.

The two-phase simplex method

Phase I:

① By multiplying some of the constraints by -1, change the problem so that $b \geq 0$

② Introduce artificial variables $y_1, \dots, y_m$ if necessary, and apply the simplex method to the auxiliary problem with cost $\sum_{i=1}^{m} y_i$.

③ If the optimal cost on the auxiliary problem is positive, the original problem is infeasible and the algorithm terminates.

(4) If the optimal cost in the auxiliary problem is zero, a feasible solution to the original problem has been found. If no artificial variable is in the final basis, the artificial variables and the corresponding columns are eliminated, and a feasible basis for the original problem is available.

(5) If the ℓ^{th} basic variable is an artificial one, examine the ℓ^{th} entry of the columns $B^{-1}A_j$, $j = 1, \dots, n$. If all of these entries ~~are~~ are ze the ℓ^{th} row represents a redundant constrain and is eliminated. Otherwise, if the ℓ^{th} entry of the j^{th} column is nonzero, apply a change basis (with this entry serving as the pivot element the ℓ^{th} basic variable exits and u_j enters the basis. Repeat the operation until all artificial variables are driven out of the basis.

Phase II:

(1) Let the final basis and tableau obtained from Phase I be the initial basis and tabl for Phase II

② Compute the reduced costs of all variables for this initial basis, using the cost coefficients of the original problem.

③ Apply the simplex method to the original problem.

Notes:

The two-phase algorithm can hande all possible outcomes ⇓